

On the efficiency of heat engines at the micro-scale and below

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We investigate the thermodynamic efficiency of sub-micro-scale heat engines operating under the conditions described by over-damped stochastic thermodynamics. We prove that at maximum power the efficiency obeys for constant isotropic mobility the universal law $\eta = 2\eta_C/(4 - \eta_C)$ where η_C is the efficiency of an ideal Carnot cycle. The corresponding power optimizing protocol is specified by the solution of an optimal mass transport problem. Such solution can be determined explicitly using well known Monge–Ampère–Kantorovich reconstruction algorithms. Furthermore, we show that the same law describes the efficiency of heat engines operating at maximum work over short time periods. Finally, we illustrate the straightforward extension of these results to cases when the mobility is anisotropic and temperature dependent.

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Present-day technology makes possible high accuracy handling of colloidal micro and sub-micro particles to study experimentally thermodynamic processes out of equilibrium [7]. A recent experiment, [4] used a highly focused infra-red laser beam to confine a $3\mu\text{m}$ diameter colloidal particle into a parabolic potential of tunable stiffness. By alternately varying the stiffness of the potential and, on much faster time scales, the temperature of the solvent where the particle was suspended [4] provides a clear evidence of the possibility to construct a Stirling heat engine at the micro-scale. This experiment is ground-breaking because it opens the way of exploring the limits to which it is possible to scale down heat engines while retaining the same working principles of their macroscopic counter-parts.

In order to set the scene for the present discussion it is expedient to start by recalling the protocol governing an ideal Stirling cycle. During such a cycle, a working system undergoes a sequence of four thermodynamic processes while being alternately in contact with two equilibrium reservoirs at different temperatures. First, an isothermal heat uptake \mathcal{Q}_{in} from the environment at temperature (measured in energy units) β_h^{-1} . During this process the system expands, i.e., it converts heat into work done on the environment. Second, an isochoric (constant-volume) cooling of the environment to a temperature $\beta_c^{-1} < \beta_h^{-1}$. Third, an isothermal compression during which the working system transfers an amount \mathcal{Q}_{out} of heat to the cold reservoir. Finally, an isochoric heating of the environment back to β_h^{-1} . The neat effect of the cycle is the extraction of an amount of work \mathcal{W}_{out} from the system with efficiency measured by the ratio $\eta = \mathcal{W}_{out}/\mathcal{Q}_{in}$. General thermodynamic reasoning [12] proves that the efficiency of any heat engine cyclically working between two classical equilibrium heat baths cannot exceed Sadi Carnot’s bound

$$\eta_C = 1 - \frac{\beta_h}{\beta_c} \quad (1)$$

The bound is attained by performing a reversible cycle.

Thus, Carnot’s bound can be classically attained in the same adiabatic limit when the power output tends to zero.

Coming back to the experimental conditions implemented in [4], a distinctive property of micro and sub-micro scale heat engines is that they operate with characteristic energies of the order of thermal fluctuations [6, 15]. The state ζ_t of the working system at time t must be thus characterized by a probability distribution $\mathbf{m}(\mathbf{z}, t)d\mathbf{z} = \mathbf{P}(\mathbf{z} < \zeta_t \leq \mathbf{z} + d\mathbf{z})$ over the space Ω of admissible configurations. The expansion (compression) of the system corresponds then to the increase (decrease) of the fluctuations of the particle induced by varying the temperature of the solvent. Thermodynamic indicators such as heat and work also become fluctuating quantities. The analysis of thermodynamic cycles needs to be rephrased in the language of stochastic thermodynamics (see e.g. [28], the more recent [24] and refs. therein).

The micro-scale heat engine of [4] relies on the heat cycle theoretical model [26] to extract thermodynamic quantities from the measurements of the colloidal particle position. In [26] the dynamics of the micro-scale particle is modeled by Langevin–Smoluchowski (over-damped) equations. Quantitative analysis of the Stirling cycle is then accomplished in the special but important case of Gaussian fluctuations. The cycle starts at time t_i when the system statistics is a normal distribution centered at the origin and with *assigned* variance σ_a^2 . Isothermal expansion at temperature β_h^{-1} is enacted by requiring that at a further time t_o^- the system probability be still normally distributed with zero average but with *assigned* variance $\sigma_b^2 > \sigma_a^2$. Then the probability remains continuous across the discontinuity of the temperature profile which at $t = t_o$ drops to β_c^{-1} . Isothermal compression brings back the variance of the distribution to σ_a^2 at $t = t_f^-$. Finally, the cycle closes at $t = t_f$ with an abrupt temperature transition from β_c^{-1} to β_h^{-1} . In order to yield a finite power output the cycle operates in a finite period $\mathcal{T} = t_f - t_i$. Hence a natural definition of

optimal efficiency [8] is that of the protocol maximizing the power output over the duration of the isothermal expansions and the period \mathcal{T} . The result for the Gaussian cycle is [26]

$$\eta_\star = \frac{2\eta_C}{4 - \eta_C} \quad (2)$$

More qualitative considerations led [26] to suggest for non-Gaussian statistics and anisotropic temperature-dependent mobility

$$\eta_\star = \frac{\eta_C}{2 - \alpha\eta_C} \quad (3)$$

with $0 \leq \alpha \leq 1$ admitting an explicit expression only in the case of isotropic mobility and recovering (2) for constant mobility. In order to compare these predictions with experiments, two questions were left open. First, to derive in the most general case the value of α in terms of the parameters of the over-damped dynamics and, second and most importantly, to prove the *realizability* of (2) and (3). This means to determine the protocols for which (2), (3) are attained. A theoretical effort in this direction was explicitly demanded by the authors of the micro-scale heat engine [6].

In the present contribution we address both questions. We specify in formula (32) below the value of the constant α appearing in (3) in terms of the mobility for any reasonable probability distributions describing the state of the system at the end of the isothermal processes. Furthermore, we prove that power optimizing protocol is specified by the well-known Monge–Ampère–Kantorovich mass reconstruction algorithm [3, 5]. To neaten the notation, we report in details the calculations in the case of constant isotropic mobility, the extension to the general case being straightforward.

We prove also a second universality result for the efficiency. It was noticed in [18] that the efficiency at maximum power for a Stirling cycle well approximates also the efficiency at maximum work output in the limit of short duration of the cycle even for ideal heat cycles other than Stirling’s. Indeed, we show that if we optimize the work output with respect to the “expansion” state \mathbf{m}_o and to the duration of the isothermal expansion then, in the limit of short cycle period ($\mathcal{T} \downarrow 0$), the efficiency tends to (3). We expect this result to be relevant for the design of nano-scale engines when controlling “target expansion” state \mathbf{m}_o may not be feasible.

Let us briefly explain the origin of the universality of our results. It stems from the fact that the entropy production during an arbitrary thermodynamic transition is proportional to the kinetic energy associated to the current velocity [23] of the system. This fact was known for some time (see e.g. [14]). Its central role, however, in mapping optimal control of stochastic thermodynamic transitions into optimal deterministic mass transport [9, 29] was only recently understood [2] (see

also [21]) and subsequently applied to predict refinements to the second law of thermodynamics and to Landauer’s bound [1] which have found experimental confirmations see e.g. [7].

Model. We suppose that the configuration space Ω of working system is \mathbb{R}^d , while its state evolves according to the Langevin–Smoluchowski dynamics

$$d\zeta_t = -\partial_{\zeta_t} U(\zeta_t, t) dt + \sqrt{\frac{2}{\beta_t}} d\omega_t \quad (4)$$

The system is driven by the gradient of a function $U: \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$ which we take in the class of confining potentials sufficiently regular to justify the manipulations which follow. Let us also start by taking for β_t^{-1} a regularized, \mathcal{T} -periodic and differentiable, version of the temperature time profile used in [26]. According to the general framework of stochastic thermodynamics [25, 27], the mean heat dissipated by the working system during a temperature cycle is equal to the expectation value of the *Stratonovich stochastic integral*

$$\mathcal{Q} = -\mathbb{E} \int_{t_i}^{t_f} d\zeta_t \cdot \partial_{\zeta_t} U(\zeta_t, t) \quad (5)$$

Well known properties of the Stratonovich integral yield after elementary manipulations [2]

$$\begin{aligned} \mathcal{Q} &= \beta_{t_i}^{-1} \mathcal{S}_{t_i} - \beta_{t_f}^{-1} \mathcal{S}_{t_f} + \\ &\mathbb{E} \int_{t_i}^{t_f} dt \left\{ \|\mathbf{v}(\zeta_t, t)\|^2 + S(\zeta_t, t) \frac{d}{dt} \frac{1}{\beta_t} \right\} \end{aligned} \quad (6)$$

where $\mathbf{v}(\mathbf{z}, t) = -\partial_{\mathbf{z}} (U(\mathbf{z}, t) + \beta_t^{-1} S(\mathbf{z}, t))$ is the current velocity of the system, $S(\mathbf{z}, t) = -\ln[\mathbf{m}(\mathbf{z}, t)/C]$ the microscopic entropy and $\mathcal{S}_t = \mathbb{E} S(\zeta_t, t)$. C is a dimensional constant irrelevant for the considerations which follow. From (6) we immediately see that if $\beta_{t_i} = \beta_{t_f}$, $\mathbf{m}(\mathbf{z}, t_i) = \mathbf{m}(\mathbf{z}, t_f)$ the only contribution to the dissipated heat comes from the entropy production. Owing to the non-degeneracy of the noise in the over-damped approximation, we are free to control the entropy production in terms of the current velocity \mathbf{v} . Optimal control strategies are then most conveniently found by searching for extremals with respect to \mathbf{m} and \mathbf{v} of the Pontryagin–Bismut functional [17]

$$\begin{aligned} \mathcal{A} &= \mathbb{E}[V(\zeta_{t_f}, t_f) - V(\zeta_{t_i}, t_i)] + \\ &\mathbb{E} \int_{t_i}^{t_f} dt \left\{ \|\mathbf{v}(\zeta_t, t)\|^2 + S(\zeta_t, t) \frac{d}{dt} \frac{1}{\beta_t} - (\mathfrak{D}V)(\zeta_t, t) \right\} \end{aligned} \quad (7)$$

The co-state function $V: \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$ plays here the role of a Lagrange multiplier imposing that

$$(\mathfrak{D}V)(\mathbf{z}, t) = [\partial_t + \mathbf{v}(\mathbf{z}, t) \cdot \partial_{\mathbf{z}}] V(\mathbf{z}, t) \quad (8)$$

acts on scalars as a total derivative along the flow generated by \mathbf{v} .

Efficiency at maximum power. We construct the optimal cycle by optimizing (6) *separately* in $[t_i, t_o]$ and $[t_o, t_f]$ under the boundary conditions $\mathbf{m}(\mathbf{z}, t_i) = \mathbf{m}(\mathbf{z}, t_f) = \mathbf{m}_i(\mathbf{z}) = C \exp\{-S_i(\mathbf{z})\}$ and $\mathbf{m}(\mathbf{z}, t_o) = \mathbf{m}_o(\mathbf{z}) = C \exp\{-S_o(\mathbf{z})\}$. We choose S_o and S_i such that the probability densities are smooth, and that the Gibbs-Shannon entropy

$$S_{t_o} - S_{t_i} \equiv S_o - S_i \equiv \mathbb{E}[S_o(\zeta_{t_o}) - S_i(\zeta_{t_i})] \geq 0 \quad (9)$$

to signify an expansion of the system. In the limit of abrupt temperature changes, the derivative of the temperature vanishes except for Dirac- δ contributions localized at t_o and t_f . Proceeding as in [1, 2], we find that along the isothermal branches of the cycle the optimal current velocity satisfies the two equations

$$\mathfrak{D}\mathbf{v} = 0 \quad \& \quad \mathfrak{D}S - \partial_{\mathbf{z}} \cdot \mathbf{v} = 0 \quad (10)$$

complemented by the stationarity condition $\partial_{\mathbf{z}} V = 2\mathbf{v}$. The set of these three equations define the Monge–Ampère–Kantorovich optimal mass transport problem [9, 29]. Let us focus on the sub-interval $[t_i, t_o]$. For any $t \in [t_i, t_o]$ the velocity satisfies

$$\mathbf{v}(\phi(t; \mathbf{z}, t_i), t_i) = \mathbf{v}(\mathbf{z}, t_i) \equiv \frac{1}{2} \partial_{\mathbf{z}} V(\mathbf{z}, t_i) \quad (11)$$

with free streaming characteristics

$$\phi(t; \mathbf{z}, t_i) = \mathbf{z} + \frac{1}{2} \partial_{\mathbf{z}} V(\mathbf{z}, t_i)(t - t_i) \quad (12)$$

The initial velocity is on its turn determined by the solution of the boundary problem

$$S(\mathbf{z}, t_o) = S(\phi(t_o; \mathbf{z}, t_i), t_o) - \text{tr} \ln[\partial_{\mathbf{z}} \otimes \phi(t_o; \mathbf{z}, t_i), t_o] \quad (13)$$

In writing (13) we imposed the continuity in time of the probability measure and of the Lagrangean map (12) at t_o . The crucial observation is that since $S(\mathbf{z}, t_o) = S_i(\mathbf{z})$, $S(\phi(t_o; \mathbf{z}, t_i), t_o) = S_o(\phi(t_o; \mathbf{z}, t_i))$ are assigned, the *Lagrangian map* $\phi(t_o; \mathbf{z}, t_i) = \phi_{\star}(\mathbf{z})$ depends only on the boundary conditions and *not* on the duration of the isothermal process. As we can repeat the same considerations for the optimization in $[t_o, t_f]$, we arrive after elementary manipulations (see e.g. [1] for details) at the general expression of the heat dissipated over one cycle

$$\mathcal{Q} = \left(\frac{1}{\beta_h} - \frac{1}{\beta_i} \right) (S_i - S_o) + \frac{\gamma(1-\gamma)}{\mathcal{T}} \mathcal{K} \quad (14)$$

For convenience we introduced the ratio $\gamma = (t_o - t_i)/\mathcal{T} \in [0, 1]$ and defined

$$\mathcal{K} = \mathbb{E} \|\phi_{\star}(\zeta_{t_i}) - \zeta_{t_i}\|^2 \geq 0 \quad (15)$$

From the mathematical slant, \mathcal{K} is the squared Wasserstein distance between the probability measures specified

by $\mathbf{m}_i(\mathbf{z})$ and $\mathbf{m}_o(\mathbf{z})$ [13]. Following [26] we define the heat input during the cycle as

$$\mathcal{Q}_{in} = \frac{S_o - S_i}{\beta_h} - \frac{\mathcal{K}}{\gamma \mathcal{T}} \quad (16)$$

and the heat output

$$\mathcal{Q}_{out} = -\frac{S_o - S_i}{\beta_c} - \frac{\mathcal{K}}{(1-\gamma) \mathcal{T}} \quad (17)$$

The definitions imply $-\mathcal{Q} = \mathcal{Q}_{in} + \mathcal{Q}_{out} = \mathcal{W}_{out}$ where \mathcal{W}_{out} is the work output. The power of the cycle is then

$$\wp \equiv \frac{\mathcal{W}_{out}}{\mathcal{T}} = \eta_C \frac{S_o - S_i}{\beta_h \mathcal{T}} - \frac{\mathcal{K}}{\gamma(1-\gamma) \mathcal{T}^2} \quad (18)$$

and its efficiency

$$\eta = \frac{\mathcal{W}_{out}}{\mathcal{Q}_{in}} = 1 - \frac{(1-\eta_C)(S_o - S_i) + \frac{\beta_h \mathcal{K}}{(1-\gamma) \mathcal{T}}}{S_o - S_i - \frac{\beta_h \mathcal{K}}{\gamma \mathcal{T}}} \quad (19)$$

having used $1 - \eta_C \equiv \beta_h \beta_c^{-1}$. The maximum power is then attained for $\gamma_{\star} = 1/2$ and

$$\mathcal{T}_{\star} = \frac{8 \beta_h \mathcal{K}}{\eta_C (S_o - S_i)} \quad (20)$$

and it is equal to

$$\wp_{\star} = \frac{\eta_C^2 (S_o - S_i)^2}{16 \mathcal{K} \beta_h^2} \quad (21)$$

Finally evaluating the efficiency at maximum power $\gamma = \gamma_{\star}$, $\mathcal{T} = \mathcal{T}_{\star}$ yields (2) independently of \mathbf{m}_i and \mathbf{m}_o . The protocol attaining the maximum power is specified by (13), which can be numerically solved using the algorithms given in [3, 5]. This is the first of our announced universality results.

Efficiency at maximum work output. We now consider a different optimization setting. As before, we allow the temperature to change only at t_o and t_f . Also as before we assign $\mathbf{m}(\mathbf{z}, t_i) = \mathbf{m}(\mathbf{z}, t_f) = \mathbf{m}_i(\mathbf{z}) = C \exp\{-S_i(\mathbf{z})\}$ and we wish to optimize with respect to the duration of the isothermal processes. The difference is now that we look for minima of the dissipated heat (equivalently, maxima of the work output) with respect to $\mathbf{m}(\mathbf{z}, t_o)$.

In such a case, the optimal current velocity still obeys (10) except when the isochoric temperature changes occur. Upon requiring the continuity of the probability density at t_o we find

$$\mathbf{v}(\mathbf{z}, t_o) - \mathbf{v}(\mathbf{z}, t_o^-) = \eta_C \frac{(\partial_{\mathbf{z}} S)(\mathbf{z}, t_o)}{2 \beta_h} \quad (22)$$

We can then avail us of the continuity of the current velocity characteristics at t_o , their free streaming form along isothermal processes, and of (11) to derive the extremal condition for the value of the initial velocity

$$\mathbf{v}(\mathbf{z}, t_i) = (1-\gamma) \eta_C \frac{(\partial_{\phi} S)(\phi(t_o; \mathbf{z}, t_i), t_o)}{2 \beta_h} \quad (23)$$

Combining this equation with (13) we finally get into a closed equation for the Lagrangian map $\varphi(\mathbf{z}) \equiv \phi(t_o; \mathbf{z}, t_i)$ governing the maximum work output

$$\frac{2\beta_h(\partial_{\mathbf{z}} \otimes \varphi)}{\gamma(1-\gamma)\eta_C} \cdot \frac{\varphi - \mathbf{z}}{\mathcal{T}} - \partial_{\mathbf{z}} \text{tr} \ln(\partial_{\mathbf{z}} \otimes \varphi) = \partial_{\mathbf{z}} S_i \quad (24)$$

We complement this equation with the boundary condition that φ maps the support of the initial density into itself ($\varphi(\mathbb{R}^d) = \mathbb{R}^d$). We emphasize that the stationarity condition $2\mathbf{v} = \partial_{\mathbf{z}} V$ implies that the Lagrangian map φ is itself a gradient map. Hence, under suitable regularity hypotheses, we have reason to expect that the problem (24) is well posed [9]. We also observe that $\partial_{\gamma} \varphi = 0$ generically solves the variational equation obtained by differentiating (24) with respect to γ if $\gamma = 1/2$. In other words, solutions of (24) attain stationarity with respect to t_o if the isothermal expansion takes half of the cycle period. This result is intuitive in light of the fact that we could optimize the work output by first looking for the optimal duration of the expansion for fixed target state at t_o and then search for the optimal state. Finally, differentiating (24) with respect to \mathcal{T} indicates that we cannot generically expect optimization with respect to \mathcal{T} to be possible. We therefore conclude that the solution φ_* of (24) describing maximum work output is obtained at $\gamma = 1/2$ and depends parametrically upon \mathcal{T} .

We now are interested in solving (24) in the limit of very short cycle period. As (12) holds up to t_o , we obtain for \mathcal{T} and \mathbf{z} sufficiently small

$$\varphi_* \approx \mathbf{z} + \frac{\eta_C \mathcal{T}}{8\beta_h} \partial_{\mathbf{z}} S_i \quad (25)$$

If the initial value of the microscopic entropy S_i confines fluctuations in the bulk region where (25) holds, we are then in the position to compute the efficiency of the cycle in short period limit. Namely, straightforward manipulations show that also in the present case the heat input and output are respectively amenable to the form (16) and (17). Using (25) we obtain within accuracy the expression of the Shannon-Gibbs entropy

$$\begin{aligned} & \mathbb{E} \text{tr} \ln \left(\partial_{\zeta_{t_i}} \otimes \varphi_* \right) (\zeta_{t_i}) \\ & \approx \frac{\eta_C \mathcal{T}}{8\beta_h} \int_{\mathbb{R}^d} d^d \mathbf{z} C e^{-S_i} \partial_{\mathbf{z}}^2 S_i = \frac{\eta_C \mathcal{T}}{8\beta_h} \mathbb{E} \|\partial_{\zeta_{t_i}} S_i\|^2 \end{aligned} \quad (26)$$

Similarly, we obtain for the squared Wasserstein distance the estimate

$$\mathcal{K} \approx \left(\frac{\eta_C \mathcal{T}}{8\beta_h} \right)^2 \mathbb{E} \|\partial_{\zeta_{t_i}} S_i\|^2 \quad (27)$$

Upon inserting these results into the expression of the efficiency we obtain

$$\lim_{\mathcal{T} \downarrow 0} \eta_* = \frac{2\eta_C}{4 - \eta_C} \quad (28)$$

We have therefore validated in the framework of over-damped stochastic thermodynamics the result of [18].

Accordingly, the efficiency of work at maximum output can be approximated for small cycle period and small temperature ratio by same formula describing efficiency at maximum power.

Elementary Extensions. In principle, anisotropic effects in the solvent may require to consider

$$d\zeta_t = -\mathbf{M}_t \cdot \partial_{\zeta_t} U(\zeta_t, t) dt + \sqrt{\frac{2\mathbf{M}_t}{\beta_t}} \cdot d\omega_t \quad (29)$$

Here \mathbf{M}_t is a strictly positive definite symmetric tensor modeling the mobility of the solvent. For many experimental applications, it is adequate to assume that the mobility tensor is constant in space and depends on time only because of changes in the solvent temperature: $\mathbf{M}_t \equiv \mathbf{M}(\beta_t)$. Under these hypotheses, we can repeat step-by-step the above calculations. The relevant indicators along isothermal transformations become the square Wasserstein distances :

$$\mathcal{K}_\ell = \mathbb{E} \{ (\phi_*(\zeta_{t_i}) - \zeta_{t_i}) \cdot \mathbf{M}_\ell^{-1} \cdot (\phi_*(\zeta_{t_i}) - \zeta_{t_i}) \} \quad (30)$$

where $\mathbf{M}_\ell \equiv \mathbf{M}(\beta_\ell)$, $\ell = h, c$ and ϕ_* is either the solution of the Monge–Ampère–Kantorovich equation or of the counter-part of (24):

$$\frac{2\beta_h(\partial_{\mathbf{z}} \otimes \varphi)}{\eta_C \gamma(1-\gamma)} \cdot \mathbf{C} \cdot \frac{\varphi - \mathbf{z}}{\mathcal{T}} - \partial_{\mathbf{z}} \text{tr} \ln(\partial_{\mathbf{z}} \otimes \varphi) = \partial_{\mathbf{z}} S \quad (31)$$

for $\mathbf{C} = \gamma \mathbf{M}_c^{-1} + (1-\gamma) \mathbf{M}_h^{-1}$. In both cases, we obtain a generalization of (2) and (28) amenable to the form (3) with

$$0 \leq \alpha = \frac{\sqrt{\mathcal{K}_h}}{\sqrt{\mathcal{K}_h} + \sqrt{\mathcal{K}_c}} \leq 1 \quad (32)$$

Using the lower and upper bounds for α , we readily recover the inequalities

$$\frac{\eta_C}{2} \leq \eta_* \leq \frac{\eta_C}{2 - \eta_C} \quad (33)$$

derived in [11] by assuming a time asymptotic behavior of the entropy production along isothermal processes inversely proportional to their duration. This is not surprising because the entropy production by optimal protocols in over-damped thermodynamics exactly satisfies such condition [1, 13].

Conclusions and Perspectives. To summarize, we derived the explicit expression (3) and (32) of the efficiency at maximum power for the Stirling heat cycle in the framework of over-damped (Langevin–Smoluchowski) dynamics. We also determined the algorithm for computing the protocols achieving such efficiency. Present day technological advances occur at an impressive speed. Laboratory evidences of the possibility of implementing Stirling cycles at the *nano*-scale is already documented in the literature [16] and the design of a possible experiment has been recently proposed

[10], see also [19, 20]. These advances pose the challenge to repeat the present analysis in the under-damped (Langevin–Kramers) regime. We recently proposed a general theory of optimal control of the dissipated heat for nano-mechanical systems governed by a Langevin–Kramers dynamics [22]. There we showed that the over-damped approximation is not only useful for perturbative analysis of the Langevin–Kramers but also provides a priori lower bounds on the dissipated heat. Based on these results, we expect that the present contribution will be relevant also for the analysis of nano-scale cyclic heat engines.

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